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METHODS OF NONLINEAR MECHANICS APPLICABLE TO VAN DER POL'S EQUATION

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METHODS OF NONLINEAR MECHANICS APPLICABLE TO VAN DER POL'S EQUATION

A THESIS
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I. Introduction

For nearly a century the mathematicians and physicists have grappled with nonlinear ordinary differential equations. Sir William Thomson and his brother, James Thomson, devised mechanisms to solve differential equations which they could not integrate. Rayleigh encountered nonlinear equations in his studies of sound, while Lindstedt and Poincaré were making the first important advances in the solution of nonlinear equations of astronomy. The twenty years following Poincaré saw no apathy toward this field, but with the advent of the electron tube and electron tube circuits, many new nonlinear equations were encountered. In 1927, B. van der Pol published a paper [1] on one of these equations, which sufficed to rekindle the general interest in nonlinear mechanics and afforded him a permanent place of prominence in the field. Since this turning point a tremendous volume of work with nonlinear equations has been published by a host of American, European and Russian mathematicians.

In spite of the emphasis of the past twenty years, there has been little progress in analytically solving nonlinear equations other than those which exhibit only slight departures from an associated linear equation. Today the majority of nonlinear equations must be solved by graphical or mechanical means or be integrated numerically by the digital computers.

Most of the interest has centered on various special cases of the second order equation

$$m\ddot{x} + \phi(x,\dot{x}) + \psi(x) = P(t),$$

where the damping force, $-\phi(x,\dot{x})$, the restoring force, $-\psi(x)$, and the exciting force, $P(t)$, can be nonlinear functions. (The dots are used to denote differentiation with respect to time.) In this paper, we shall consider methods which are applicable to free systems with nonlinear damping forces, which are represented by

(1.1) $\quad m\ddot{x} + \phi(x,\dot{x}) + kx = 0.$

As an example, each method will be applied to the van der Pol equation,

$$m\ddot{x} - (\epsilon\dot{x} - \frac{\beta}{3}\dot{x}^3) + kx = 0.$$

Since the transformation $t_1 = \sqrt{k/m}\, t$ and $x_1 = \sqrt{3\beta/\epsilon}\, x$ brings this into the form

$$\frac{d^2x_1}{dt_1^2} - \frac{\epsilon}{\sqrt{mk}}\left[\frac{dx_1}{dt_1} - \frac{1}{3}\left(\frac{dx_1}{dt_1}\right)^3\right] + x_1 = 0$$

we shall use only the simplified equation

(1.2) $\quad \ddot{x} - \epsilon(\dot{x} - \frac{1}{3}\dot{x}^3) + x = 0.$

II. Qualitative Methods

Poincaré's greatest accomplishments in nonlinear mechanics were methods of discussing the nature of the solution of an equation rather than of actually finding the solution [2]. In practical problems, a qualitative knowledge of the solution is frequently sufficient. For instance,

the engineer who can deduce that the vibrations of a contemplated structure will be small, is not interested in the exact form of these vibrations.

By considering $x = v$, a separate variable, we can reduce (1.1) to the first order equation

$$(2.1) \qquad \frac{dv}{dx} = -\frac{\phi(x,v) + kx}{mv}$$

or the parametric system

$$(2.2) \qquad \begin{aligned} \dot{v} &= -\left[\phi(x,v) + kx\right] \\ \dot{x} &= mv \end{aligned}$$

Equations (2.2) determine a field of directions which are followed by the solution or integral curves in the x,v plane. If we first consider the case for $\phi \equiv 0$, we find that the solution curves are the one-parameter family of ellipses

$$mv^2 + kx^2 = c^2$$

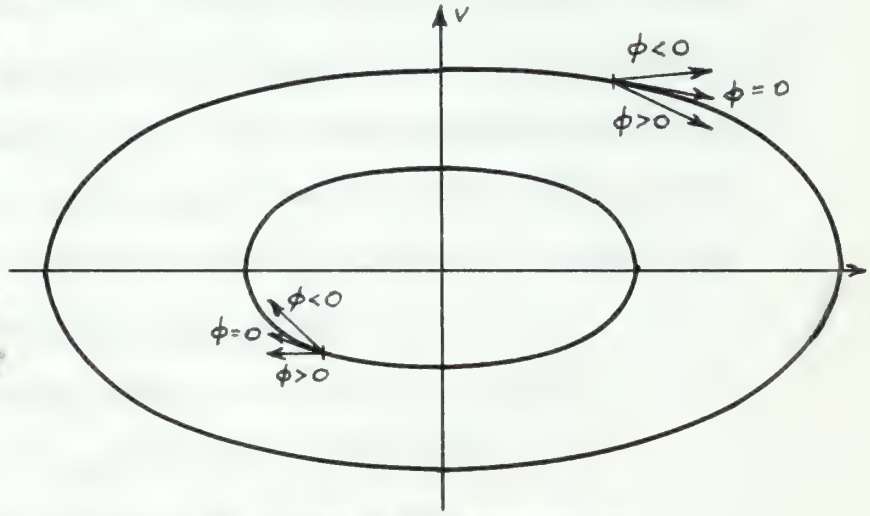

Fig. 1 Effect of damping on field direction.

Figure 1. Examining (2.2), we see that the field vector at any point will tend toward the inside or toward the outside of the ellipse passing through that point, depending on the sign of $\phi(x,v)$. This effect is shown for two points in Figure 1. By thus considering the

field directions at a number of points, we can obtain an idea as to the shape and extent of the solution curves.

For our example, we have

$$\dot{v} = \epsilon(v - \tfrac{1}{3}v^3) - x$$
$$\dot{x} = v \tag{2.2}$$

From these we see that the solution curves for $\epsilon = 0$ are the circles

$$x^2 + v^2 = c^2,$$

Figure 4. For $\epsilon \neq 0$ and $0 < v^2 < 3$, the field directions tend toward the outside of the circles; while for $4 < v^2$, the field directions tend toward the inside. From this we may reason that small oscillations increase and large oscillations decrease. This suggests that there might exist some region of medium amplitude toward which all oscillations tend. By drawing the field directions throughout this region, we see that there is an irregular annular region, R, bounded by c_1 and c_2, with the property that the field directions at every point on the boundary tend toward the interior of the ring. In order to better understand the significance of this, we quote a theorem of Poincaré and Bendixson:

> "If a solution curve, $x = x(t)$, $y = y(t)$, of $dy/dx = P(x,y)/Q(x,y)$, with P and Q defined and regular in $-\infty < x,y < \infty$, remains within a bounded region of the x,y plane without approaching singular points, then there exists at least one closed solution curve of the differential equation."

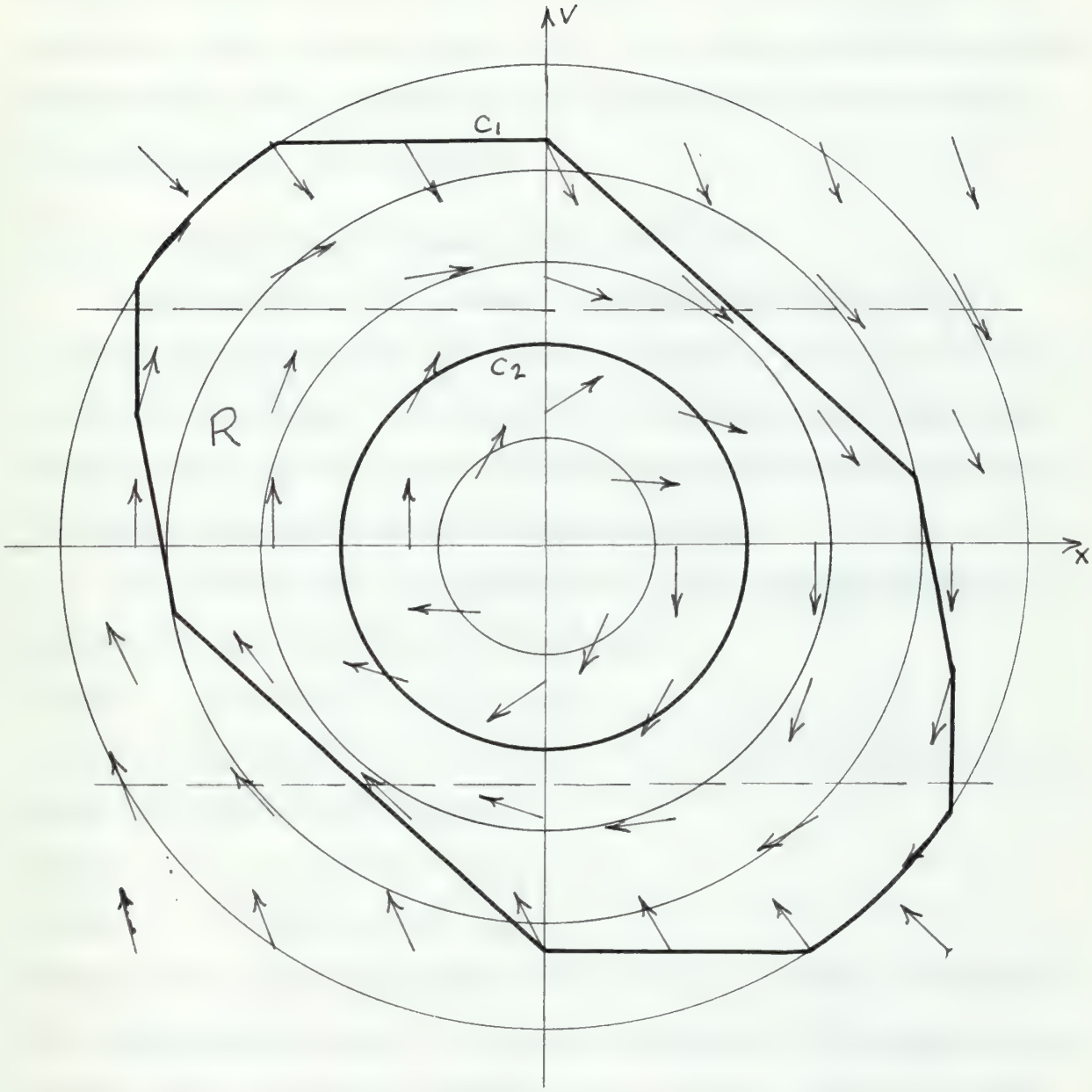

Fig. 9 Field directions of solution curves for van der Pol's equation for $\mu = 1$.

Equations (2.3) and 5 satisfy this theorem, and therefore the van der Pol equation has a closed or periodic solution, called a limit cycle, lying in A. All the particular solutions will enter A and will therefore approach this limit cycle as time increases indefinitely.

III. The Method of Kryloff and Bogoliuboff

This method [1], commonly referred to as the K.B. method, is essentially the same as Lagrange's method of "variation of parameters" as adapted for use with nonlinear equations by van der Pol, who utilized only the leading terms of a Fourier expansion of the nonlinear term.

The method will be developed to find a first approximation of the solution of the equation

(3.1) $\ddot{x} + \epsilon f(x,\dot{x}) + u^2 x = 0,$

where we assume that ϵ is small. For $\epsilon = 0$, we already know that (3.1) has the solution

(3.2) $x = a \sin (ut + \phi),$

(3.3) $\dot{x} = au \cos (ut + \phi)$

where a and ϕ are constants. For the K.B. method, we assume that the solution for $\epsilon \neq 0$ can be expressed in the same form, where a and ϕ are now functions of ϵ and t. Following this assumption and differentiating (3.2), we find

$$\dot{x} = au \cos (ut+\phi) - \dot{a} \sin (ut+\phi) + a\dot{\phi} \cos (ut+\phi).$$

Since (3.3) must hold, we conclude that

(3.4) $\dot{a} \sin (ut + \phi) + a\dot{\phi} \cos (ut + \phi) = 0.$

If we differentiate (3.3), we have

$$(3.5)\qquad \ddot{x} = -au^2\sin(ut+\phi) + \dot{a}u\cos(ut+\phi) - au\dot{\phi}\sin(ut+\phi),$$

and substituting (3.3), (3.3), and (3.5) into (3.1), we have

$$(3.6)\qquad \dot{a}u\cos(ut+\phi) - au\dot{\phi}\sin(ut+\phi) = -\epsilon f\left[a\sin(ut+\phi), au\cos(ut+\phi)\right].$$

Then if we solve (3.4) and (3.6) simultaneously, we have

$$(3.7)\qquad \dot{a} = -\frac{\epsilon}{u} f\left[a\sin(ut+\phi), au\cos(ut+\phi)\right],$$

$$(3.8)\qquad \dot{\phi} = \frac{\epsilon}{au} f\left[a\sin(ut+\phi), au\cos(ut+\phi)\right].$$

We can expand the nonlinear term in the Fourier series

$$(3.9)\qquad f\left[a\sin(ut+\phi), au\cos(ut+\phi)\right] = f_0 + \sum_{n=1}^{\infty}\left[f_n\cos n(ut+\phi) + g_n\sin n(ut+\phi)\right],$$

where the coefficients are functions of a. Now if we substitute (3.8) into (3.7) and (3.8), apply trigonometric identities and gather terms, we have

$$(3.10)\qquad \dot{a} = -\frac{\epsilon}{u}\Big\{\tfrac{1}{2}f_1 + (f_0 + \tfrac{1}{2}f_2)\cos(ut+\phi) + \tfrac{1}{2}g_2\sin(ut+\phi) + \tfrac{1}{2}\sum_{n=2}^{\infty}\left[(f_{n-1} + f_{n+1})\cos n(ut+\phi) + (g_{n-1} + g_{n+1})\sin n(ut+\phi)\right]\Big\},$$

$$(3.11)\qquad \dot{\phi} = \frac{\epsilon}{au}\Big\{\tfrac{1}{2}g_1 + (g_0 - \tfrac{1}{2}g_2)\cos(ut+\phi) + \tfrac{1}{2}f_2\cos(ut+\phi) + \tfrac{1}{2}\sum_{n=2}^{\infty}\left[(g_{n-1} - g_{n+1})\cos n(ut+\phi) + (f_{n+1} - f_{n-1})\cos n(ut+\phi)\right]\Big\}.$$

Since ϵ is a small parameter, $\dot{a}$ and $\dot{\phi}$ are also small; and, in order to obtain a first approximation, we can take a and ϕ to be constants during integration over any one period $T = \frac{2\pi}{u}$. If we integrate (3.10) and (3.11) over the interval from t to t + T, the trigonometric terms all vanish, leaving

$$\int_t^{t+T} \dot{a}\, dt = \int_t^{t+T} \left(-\frac{\epsilon f_1}{2u}\right) dt$$

$$a(t+T) - a(t) = -\frac{\epsilon f_1}{2u} T$$

$$\frac{a(t+T)-a(t)}{T} = -\frac{\epsilon f_1}{2u}$$

and

$$\int_t^{t+T} \dot{\phi}\, dt = \int_t^{t+T} \left(\frac{\epsilon g_1}{2au}\right) dt$$

$$\phi(t+T) - \phi(t) = \frac{\epsilon g_1}{2aT}$$

$$\frac{\phi(t+T)-\phi(t)}{T} = \frac{\epsilon g_1}{2au}.$$

We then replace the difference quotients above by the derivatives to obtain

$$(3.12) \qquad \dot{a} = -\frac{\epsilon f_1}{2u}$$

$$(3.13) \qquad \dot{\phi} = \frac{\epsilon g_1}{2au}.$$

These are the differential equations of the first approximations of a and ϕ. If we denote the quantities thus found by a_0 and ϕ_0, the first approximation of the solution of (3.1) is

$$x_0 = a_0 \sin(ut+\phi_0).$$

To check the accuracy of x_0, we substitute it into (3.1) and find that the solution is in error for terms of the order of ϵ. This leaves it highly desirable to improve our first approximation. We shall find a refined first approximation,

called

$$x^* = a^* \sin(ut + \phi^*),$$

by taking into account the more rapidly varying parts of $\dot{a}$ and $\dot{\phi}$ which are given by the trigonometric terms, and shall represent a^* and ϕ^* in the form

$$a^* = a_0 + \epsilon a_1 ,$$

$$\phi^* = \phi_0 + \epsilon \phi_1 .$$

By use of a method of successive approximations in which the a and ϕ in the trigonometric terms of (2.10) and (2.11) are taken as a_0 and ϕ_0, we can find x^* with accuracy of terms of the order of ϵ. By substituting (2.12) and (2.13) into (2.10) and (2.11), we have

$$(2.14) \qquad \dot{a}^* = \dot{a}_0 - \frac{\epsilon}{2}\Big\{(f_1 + \tfrac{1}{2}g_2)\cos(ut+\phi) + \tfrac{1}{2}g_2 \sin(ut+\phi_0) + \frac{1}{2}\sum_{n=0}^{\infty}\Big[(f_{n-1}+f_{n+1})\cos n(ut+\phi_0) + (g_{n-1}+g_{n+1})\sin n(ut+\phi)\Big]\Big\},$$

$$(2.15) \qquad \dot{\phi}^* = \dot{\phi}_0 + \frac{\epsilon}{2a_0}\Big\{(f_1 - \tfrac{1}{2}g_2)\sin(ut+\phi_0) + \tfrac{1}{2}g_2 \cos(ut+\phi_0) + \frac{1}{2}\sum_{n=0}^{\infty}\Big[(f_{n-1}-f_{n+1})\sin n(ut+\phi_0) + (g_{n+1}-g_{n-1})\cos n(ut+\phi_0)\Big]\Big\},$$

where the coefficients are functions of a_0, and we have neglected terms of order ϵ^2. We can hold a_0 and ϕ_0 constant during integration because any variation would produce terms of order of ϵ^2 which we neglect. Integrating (2.14) and (2.15) we have

$$(2.16)\qquad a^* = a_0 - \frac{\epsilon}{\omega^2}\Big\{(c_0+\tfrac{1}{2}c_2)\sin(ut+\phi_0) - \tfrac{1}{2}c_2\cos(ut+\phi_0)$$
$$+\tfrac{1}{2}\sum_{n=2}^{\infty}\Big[\tfrac{1}{n}(c_{n-1}+c_{n+1})\sin n(ut+\phi_0) - \tfrac{1}{2}(c_{n-1}+c_{n+1})\cos n(ut+\phi_0)\Big]\Big\}$$
$$= a_0 + \epsilon a_1 \ ,$$

$$(2.17)\qquad \phi^* = \phi_0 + \frac{\epsilon}{a_0\omega^2}\Big\{-(c_0-\tfrac{1}{2}c_2)\cos(ut+\phi_0) + \tfrac{1}{2}c_2\sin(ut+\phi_0)$$
$$+\tfrac{1}{2}\sum_{n=2}^{\infty}\Big[\tfrac{1}{n}(c_{n+1}-c_{n-1})\cos n(ut+\phi_0) + \tfrac{1}{2}(c_{n+1}-c_{n-1})\sin n(ut+\phi_0)\Big]\Big\}$$
$$= \phi_0 + \epsilon\phi_1 \ .$$

We can expand x^* in a power series in ϵ to obtain

$$(2.18)\qquad x^* = a^*\sin(ut+\phi^*) = (a_0+\epsilon a_1)\sin(ut+\phi_0+\epsilon\phi_1)$$
$$= a_0\sin(ut+\phi_0) + \epsilon a_1\sin(ut+\phi_0) + \epsilon a_0\phi_1\cos(ut+\phi_0) + O(\epsilon^2)$$

Now substituting (2.16) and (2.17) into (2.18) we have

$$x^* = a_0\sin(ut+\phi_0) - \frac{\epsilon}{\omega^2}\Big\{(c_0+\tfrac{1}{2}c_2)\sin(ut+\phi_0)$$
$$-\tfrac{1}{2}c_2\cos(ut+\phi_0) + \tfrac{1}{2}\sum_{n=2}^{\infty}\Big[\tfrac{1}{n}(c_{n-1}+c_{n+1})\sin n(ut+\phi_0)$$
$$-\tfrac{1}{n}(c_{n-1}+c_{n+1})\cos n(ut+\phi_0)\Big]\Big\}\sin(ut+\phi_0)$$
$$+\frac{\epsilon}{\omega^2}\Big\{-(c_0-\tfrac{1}{2}c_2)\cos(ut+\phi_0) + \tfrac{1}{2}c_2\sin(ut+\phi_0)$$
$$+\tfrac{1}{2}\sum_{n=2}^{\infty}\Big[\tfrac{1}{n}(c_{n+1}-c_{n-1})\cos n(ut+\phi_0) + \tfrac{1}{2}(c_{n+1}-c_{n-1})\sin n(ut+\phi_0)\Big]\Big\}\cdot$$
$$\cos(ut+\phi_0) + O(\epsilon^2);$$

and, finally, if we apply trigonometric identities and gather terms, this reduces to the result

$$(2.19)\qquad x^* = a_0\sin(ut+\phi_0) + \frac{\epsilon}{\omega^2}\Big\{-c_0 + \sum_{n=2}^{\infty}\frac{1}{(n-1)}\Big[c_n\cos n(ut+\phi_0)$$
$$+ d_n\sin n(ut+\phi_0)\Big]\Big\} + O(\epsilon^2) \ .$$

This is a usable formula for the refined first approximation and can be applied to a particular equation.

We can check x^* by substituting back into (2.1). After some calculation, we find

$$(2.20)\qquad \ddot{x}^* + \omega^2 x^* = -\epsilon\left\{f_0 + \sum_{n=1}^{\infty}\left[f_n \cos n(\omega t+\phi_0) + g_n \sin n(\omega t+\phi_0)\right]\right\} + O(\epsilon^2)$$

$$= -\epsilon f\left[a_0 \sin(\omega t+\phi_0), \omega a_0 \cos(\omega t+\phi_0)\right] + O(\epsilon^2)$$

by (2.9). $f\left[a^* \sin(\omega t+\phi^*), \omega a^* \cos(\omega t+\phi^*)\right] = f(x^*, \dot{x}^*)$ can be expanded in a power series in ϵ, so that

$$(2.21)\qquad \epsilon f\left[a^* \sin(\omega t+\phi^*), \omega a^* \cos(\omega t+\phi^*)\right] =$$

$$\epsilon f\left[a_0 \sin(\omega t+\phi_0), \omega a_0 \cos(\omega t+\phi_0)\right] + O(\epsilon^2)$$

Then combining (2.20) and (2.21), we have

$$\ddot{x}^* + \epsilon f(x^*, \dot{x}^*) + \omega^2 x^* = O(\epsilon^2),$$

which confirms the intended accuracy of x^*.

To apply the K.B. method to the van der Pol equation (1.2), we set $\omega = 1$ and $f(x, \dot{x}) = -\dot{x} + \frac{1}{3}\dot{x}^3$.

Letting $\dot{x} = a_0 \omega \cos(t+\phi_0)$, we find

$$(2.22)\qquad f(x, \dot{x}) = a_0\left(\tfrac{1}{4}a_0^2 - 1\right)\cos(t+\phi_0) + \tfrac{1}{12}a_0^3 \cos 3(t+\phi_0);$$

so that $f_1 = a_0\left(\frac{1}{4}a_0^2 - 1\right)$, $f_3 = \frac{1}{12}a_0^3$, and all the other coefficients of (2.8) are zero. From (2.18),

$$\dot{a}_0 = -\frac{\epsilon f_1}{2\omega} = \frac{1}{2}\epsilon a_0\left(1 - \frac{1}{4}a_0^2\right),$$

which gives on integration,

$$(2.23)\qquad a_0 = \frac{2}{\sqrt{e^{-\epsilon t} + \frac{c^2}{4}}}$$

From (2.15),

$$\dot{\phi}_0 = \frac{\epsilon g_1}{2a_0\omega} = 0,$$

where ϕ_0 is a constant. By substituting (2.23) into (2.19) we find the approximate solution

$$(3.24) \qquad x^{*} = \frac{c}{(e^{-\epsilon t} + \frac{c^2}{4})^{\frac{1}{2}}} \sin(t+\phi_0) + \frac{\epsilon c^2}{(e^{-\epsilon t} + \frac{c^2}{4})^{\frac{3}{2}}} \cos 3(t+\phi_0) + O(\epsilon^2) .$$

We note that

$$(3.25) \qquad \lim_{t \to \infty} x^{*} = 2 \sin(t+\phi_0) + \frac{\epsilon}{2} \cos 3(t+\phi_0) + O(\epsilon^2)$$

regardless of the value of c, which is determined by the initial amplitude. This is an approximation of our limit cycle.

IV. The Perturbation Method

This is a method in which it is assumed that the solution of (1.) can be approximated by a periodic solution, x, with frequency, u, where both x and u can be expressed by power series in ϵ. Prior to 1900, a perturbation method was applied to linear equations by expanding only x in power series in ϵ, but Poincaré and his contemporaries found that this failed for nonlinear equations because of secular or nonperiodic terms which entered. Lindstedt overcame this difficulty by expanding both x and u in power series in ϵ.

In order to justify this method, it is necessary to prove that there can exist a periodic solution to (1.1) which can be expressed in a power series in ϵ. By the change of variable $t_1 = \sqrt{k/mt}$, (1.1) can be transformed to (4.1) $\ddot{x} + \epsilon f(x,\dot{x}) + x = 0$,

where we shall assume that ϵ is small. We shall prove that a periodic solution to (4.1) exists which depends analytically

on ϵ, from which the existence of a power series expansion follows immediately by Taylor's theorem. We assume that $f(x,\dot{x})$ is an analytic function of x and $\dot{x}$ for all values of these arguments. Then by the method of Lindle [6], in which a Taylor's series satisfying the differential equation is derived and a convergent continued series is constructed, we know that unique solutions exist which are analytic functions of t and ϵ throughout the region where $f(x,\dot{x})$ is analytic. In particular, we can select a solution,

$$x = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots$$

which satisfies the initial values $x(0) = A$ and $\dot{x}(0) = 0$ and which remains in the domain $|\,| \leq 2|A|$ and $|\dot{\,}| \leq 2|A|$ for $|\epsilon| \leq \epsilon_1$. This domain is selected quite arbitrarily in order to agree with a result which we will encounter later.

For $\epsilon = 0$, (4.1) reduces to the linear equation $\ddot{x} + x = 0$. This has the solution $x = A\cos t$ which satisfies the initial values and lies in the domain above. We shall establish the existence of periodic solutions $x[t, A(\epsilon), \epsilon]$ in the neighborhood of $\epsilon = 0$.

We can replace our initial value problem by the integral equation

$$(4.2) \qquad x(t, A, \epsilon) = A\cos t - \epsilon\int_0^t f[x(s), \dot{x}(s)]\sin(t-s)\,ds.$$

We will show that the two are equivalent and that solution of (4.2) satisfies the initial value problem and that solution of the initial value problem satisfies (4.2). Differentiating (4.2), we have

$$(4.7) \qquad \dot{x}(t,A,\epsilon) = -A\sin t - \epsilon\int_0^t f\left[x(s),\dot{x}(s)\right]\cos(t-s)\,ds$$

and

$$\ddot{x}(t,A,\epsilon) = -A\cos t + \epsilon\int_0^t f\left[x(s),\dot{x}(s)\right]\sin(t-s)\,ds$$
$$- \epsilon f\left[x(t),\dot{x}(t)\right],$$

so that

$$\ddot{x}(t,A,\epsilon) + x(t,A,\epsilon) = -\epsilon f\left[x(t),\dot{x}(t)\right].$$

Examining (4.3) and (4.5), we see that $x(0,A,\epsilon) = A$ and $\dot{x}(0,A,\epsilon) = 0$, which confirms that the solution of (4.3) satisfies the initial value problem. To show the converse, let $z = x + y$, where

$$y = \epsilon\int_0^t f\left[x(s),\dot{x}(s)\right]\sin(t-s)\,dt.$$

If we differentiate twice, we find

$$\ddot{y} = -\epsilon\int_0^t f\left[x(s),\dot{x}(s)\right]\sin(t-s)\,dt - \epsilon f\left[x(t),\dot{x}(t)\right],$$

so that

$$\ddot{y} + y = \epsilon f\left[x(t),\dot{x}(t)\right].$$

From (4.1), we already know that

$$\ddot{x} + x = -\epsilon f\left[x(t),x(t)\right],$$

therefore

$$\ddot{z} + z = (\ddot{x} + \ddot{y}) + (x + y)$$
$$= (\ddot{x} + x) + (\ddot{y} + y) = 0.$$

We know that $\ddot{z} + z = 0$ has the solution, $z = A\cos t$ which satisfies the initial value problem; therefore

$$y = z - x = A\cos t - x$$

Finally, we see that we can substitute in our integral equation, (4.5), to obtain

$$
\begin{aligned}
x(t,A,\epsilon) &= A\cos t - \epsilon\int_0^t f\left[x(s),\dot{x}(s)\right]\sin(t-s)\,ds \\
&= A\cos t - y \\
&= A\cos t - (A\cos t - x) \\
&= x.
\end{aligned}
$$

From this, we conclude that (4.2) is satisfied by the solution of the initial value problem, which we know exists.

The integral equation (4.2) could be derived by means of the theory of Green's functions [5, Chapter II]. The function

$$
g(t,s) = \begin{cases} \sin(t-s), & 0 \leq s \leq t \\ 0, & t < s \end{cases}
$$

has the properties

1) It is continuous in the region $-\ t$,
2) Its derivative is discontinuous at $t = s$,
3) It formally satisfies (4.1) when $= 0$,
4) g and $\dot{g}$ vanish at $t = 0$;

therefore, it is the Green's function for the homogeneous case of our initial value problem and can be used to transform the differential system to an integral equation.

Continuing with our proof that a periodic solution exists in the domain $|x| \leq 3|A|$ and $|\dot{x}| \leq 3|A|$ for $|t| \leq T_1$, let M be the maximum value of $f(x,\dot{x})$ in this domain. If we select $\epsilon \leq \frac{A}{3M}$, we see by (4.2) that for $|t| \leq 3\pi$

$$
\begin{aligned}
|x(t,A,\epsilon)| &\leq |A\cos t| + \left|\epsilon\int_0^t f[x(s),\dot{x}(s)]\sin(t-s)\,ds\right| \\
&\leq |A| + \left|\frac{A}{2\pi}\right| \int_0^t M\left|\sin(t-s)\right| ds \\
&\leq |A| + \left|\frac{A}{2\pi}\right| \int_0^t ds = \left|A\left(1+\frac{t}{2\pi}\right)\right| \\
&\leq 2|A|.
\end{aligned}
$$

Similarly, we find

$$|\dot{x}(t,A,\epsilon)| \leq 2|A|,$$

which proves that our solution could remain inside the required region. If a periodic solution exists, it will satisfy

(4.4) $\quad x(T,A,\epsilon) = x(0,A,\epsilon) = A$

(4.5) $\quad \dot{x}(T,A,\epsilon) = \dot{x}(0,A,\epsilon) = 0$

where T is the period $\frac{2\pi}{\omega}$. These equations imply periodicity because they indicate that the original initial value problem is repeated at time T. Since our solutions are unique for given initial conditions, the solution commencing at time T will be identical with the one commencing at time zero and will, as a result, repeat the same initial values at time $2T$. In this manner, the solution will be repeated over each interval T. If we can select $T(\epsilon)$ and $A(\epsilon)$ to satisfy (4.4) and (4.5), our periodic solution is secured. We would like to apply the Implicit Function Theorem to show that we can solve these equations for T and A; but, since (4.4) degenerates to $1 \equiv 1$ for $\epsilon = 0$ and $T = T^*$, we must alter their form somewhat. Applying the transformation

$$T = 2\pi + \epsilon p(\epsilon)$$

$$F = \epsilon^{-1}\left[A - x(T,A,\epsilon)\right] = 0$$

$$G = \epsilon^{-1}\left[\dot{x}(T,A,\epsilon)\right] = 0$$

to (4.2) and (4.3), we obtain

$$(4.6) \qquad U = A\,\epsilon^{-1}(1-\cos p) - \int_0^{2\pi+\epsilon p} f\left[x(s),\dot{x}(s)\right]\sin(s-\epsilon p)\,ds,$$

$$(4.7) \qquad V = A\,\epsilon^{-1}\sin\epsilon p + \int_0^{2\pi+\epsilon p} f\left[x(s),\dot{x}(s)\right]\cos(s-\epsilon p)\,ds$$

$\epsilon^{-1}(1-\cos\epsilon p) = \frac{\epsilon^{0}p^{2}}{2!} - \frac{\epsilon^{2}p^{4}}{4!} + \cdots$ and $\epsilon^{-1}\sin\epsilon p = p - \frac{\epsilon^{2}p^{3}}{3!} + \cdots$ are analytic functions of ϵ in the neighborhood of $\epsilon = 0$; therefore, F and G are analytic functions of A, ϵ, and p in the same neighborhood.

Now we know, by the Implicit Function Theorem, that if we can find A and p which satisfy (4.6) and (4.7) for $\epsilon = 0$ and if the Jacobian for $\epsilon = 0$

$$(4.8) \qquad J\left(\frac{U,V}{A,p}\right) = -A\left\{\int_0^{2\pi}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial A} + \frac{\partial f}{\partial \dot{x}}\cdot\frac{\partial \dot{x}}{\partial A}\right)\sin s\,ds\right\} \neq 0,$$

that the periodicity equations can be solved in the neighborhood of $\epsilon = 0$. These conditions are sufficient to prove that a periodic solution to (4.1) exists which depends analytically on ϵ. They must be checked for the individual equations to which we wish to apply the perturbation method.

We shall now turn to our example, the van der Pol equation, (1.1), in which $f(x,\dot{x}) = -\dot{x} + \frac{1}{3}\dot{x}^3$. From (4.6) and (4.7), we have

$$P = A\epsilon^{-1}(1-\cos\epsilon p) + \int_0^{2\pi+\epsilon p} (\dot{x} - \tfrac{1}{2}\dot{x}^2)\sin(s-p)\,ds,$$

$$Q = A\epsilon^{-1}\sin\epsilon p - \int_0^{2\pi+\epsilon p} (\dot{x} - \tfrac{1}{2}\dot{x}^2)\cos(s-p)\,ds.$$

We already know that, for $\epsilon = 0$, $x = A\cos t$ and $\dot{x} = -A\sin t$; therefore, for $\epsilon = 0$,

$$P = \int_0^{2\pi} (A\sin s - \tfrac{1}{2}A^2\sin^2 s)\sin s\,ds$$

$$= \int_0^{2\pi} \left[\tfrac{1}{2}A - \tfrac{1}{2}A^2 + \tfrac{1}{2}(\tfrac{1}{2}A^2 - A)\cos 2s + \tfrac{1}{6}A^2\cos 4s\right] ds$$

$$= A\pi\,(1 - \tfrac{1}{4}A^2),$$

and

$$Q = Ap - \int_0^{2\pi} (A\sin s - \tfrac{1}{2}A^2\sin^2 s)\cos s\,ds$$

$$= Ap - \int_0^{2\pi} \left[\tfrac{1}{2}A(1-\tfrac{1}{2}A^2)\sin 2s - \tfrac{1}{6}A^2\sin 4s\right] ds$$

$$= Ap\,.$$

From these, we see that a non-trivial solution of P and Q for $\epsilon = 0$ is $A = \pm 2$ and $p = 0$. Since $f(x,\dot{x}) = -\dot{x} + \frac{1}{2}\dot{x}^2$, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial \dot{x}} = -1 + \dot{x}$. Bearing in mind that for $\epsilon = 0$, $\dot{x} = -A\sin t$, we have from (4.8)

$$J\left(\frac{\Delta,\Delta}{A,p}\right) = -A\int_0^{2\pi}(-1+A^2\sin^2 s)(-\sin s)\sin s\,ds$$

$$= A\int_0^{2\pi}\left[\frac{3A^2}{2}-\frac{1}{2}+\frac{1}{2}(1-A^2)\cos 2s+\frac{A^2}{2}\cos 4s\right]ds$$

$$= \pi A\left(\frac{3}{4}A^2-1\right) = \pm 4\pi \neq 0 .$$

This assures us that we can apply the perturbation method to the van der Pol equation.

The mechanics of carrying out the method are adequately demonstrated by applying it to our example; therefore, we will not consider the general case. Since we wish to expand both x and u in powers of ϵ, we must transform (1.2) somewhat so as to make u explicit. If we let $\theta = ut$ and denote differentiation with respect to θ by primes, (1.2) becomes

(4.9) $\quad u^2x'' - \epsilon(ux' - \frac{1}{3}u^3x'^3) + x = 0$

where $x(\theta)$ has period 2π. We will assume the power series

(4.10) $\quad x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots,$

(4.11) $\quad u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$

and the initial value $u'(0) = 0$. Differentiating and taking powers, we have

$$x' = x_0' + \epsilon x_1' + \epsilon^2 x_2' + \cdots,$$

$$x_0'^3 = x_0'^3 + (3x_0'^2x_1')\epsilon + (3x_0'x_1'^2 + 3x_0'^2x_2')\epsilon^2$$
$$+ (x_1'^3 + 3x_0'^2x_3' + 6x_0'x_1'x_2')\epsilon^3 + \cdots,$$

$$x'' = x_0'' + \epsilon x_1'' + \epsilon^2 x_2'' + \cdots,$$

$$u^2 = u_0^2 + (2u_0u_1)\epsilon + (u_1^2 + 2u_0u_2)\epsilon^2 + (2u_0u_2 + 2u_1u_2)\epsilon^3 + \cdots,$$

$$u^3 = u_0^3 + (3u_0^2u_1)\epsilon + (3u_0^2u_2 + 3u_0u_1^2)\epsilon^2$$

$$+ (u_1^3 + 3u_0u_2 + 3u_0u_1u_2)\epsilon^3 + \cdots.$$

If we substitute these into (4.9) and gather terms into powers of ϵ, we have

$$(4.10) \quad \left[u_0^2x_0'' + x_0\right] + \left[u_0^2x_1'' + 2u_0u_1x_0'' - u_0x_0' + \tfrac{1}{2}u_0^2x_0'^2 + x_1\right]\epsilon$$

$$+ \left[u_0^2x_2'' + 2u_0u_1x_1'' + u_1^2x_0'' - u_1x_0' - u_0x_1' + u_0^2u_1x_0'^2 + u_0^3x_0'x_1'\right.$$

$$\left. + x_2\right]\epsilon^2 + \left[u_0^2x_3'' + 2u_0u_1x_2'' + u_1^2x_1'' + 2u_0u_2x_1'' + 2u_0u_3x_0''\right.$$

$$+ 2u_1u_2x_0'' - u_2x_0' - u_1x_1' - u_0x_2' + u_0^2u_2x_0'^2 + u_0u_1x_0'^3$$

$$\left. + 3u_0^2u_1x_0'^2x_1' + u_0^3x_0'x_1'^2 + u_0^3x_0'^2x_2' + x_3\right]\epsilon^3 + \cdots = 0.$$

Since (4.10) is true for all small ϵ and the powers of ϵ are linearly independent, the coefficient of each power of ϵ must vanish. Furthermore, since $x(\theta)$ has period 2π and $x'(0) = 0$, each x_i must have period 2π and each $x_i'(0) = 0$. First taking $u_0^2x_0'' + x_0 = 0$, we know this has the solution

$$x_0 = A_0 \cos\frac{\theta}{u_0} + B_0 \sin\frac{\theta}{u_0}.$$

Since x_0 has period 2π and $x_0'(0) = 0$,

(4.13) $\quad B_0 = 0,\ u_0 = 1$, and $x_0 = A_0 \cos\theta$.

Next taking the coefficient of ϵ in (4.12), substituting from (4.37) and collecting terms, we have

$$x_1'' + x_1 = 2u_1 A_0 \cos\theta + \left(\frac{A_0^3}{4} - A\right)\sin\theta - \frac{A_0^3}{12}\sin 3\theta .$$

The general solution of this equation as it stands will include non-periodic terms of type $C\theta \sin\theta$ and $D\theta \cos\theta$, but we have stipulated that x_1 be periodic; therefore, the coefficients of $\cos\theta$ and $\sin\theta$ on the right side must vanish. Thus, we conclude

(4.14) $\quad A_0 = \pm 2$ and $u_1 = 0$.

We will use $A_0 = -2$ because the negative sign would amount only to an initial phase difference. This leaves

$$x_1'' + x_1 = -\frac{2}{3}\sin 3\theta ;$$

from which, we find

$$x_1 = A_1 \cos\theta + B_1 \sin\theta + \frac{1}{27}\sin 3\theta.$$

Since $x_1'(0) = 0$, $B_1 = -\frac{1}{6}$ and

(4.15) $\quad x_1 = A_1 \cos\theta - \frac{1}{6}\sin\theta + \frac{1}{27}\sin 3\theta.$

Proceeding with the process, we take the coefficient of ϵ^2 from (4.12), substitute from (4.17) - (4.19), and collect terms, to obtain

$$x_2'' + x_2 = (4u_2 + \tfrac{1}{4})\cos\theta + 2A_1 \sin\theta - A_1 \sin 3\theta$$
$$- \frac{1}{2}\cos 3\theta + \frac{1}{4}\cos 5\theta.$$

To avoid non-periodic terms in x_2, we set

(4.16) $\quad A_1 = 0$ and $u_2 = -\frac{1}{16}$,

which leaves

$$x_2'' + x_2 = -\frac{1}{3}\cos 3\theta + \frac{1}{4}\cos 4\theta ,$$

from which we find

$$x_2 = A_2\cos\theta + B_2\sin\theta + \frac{1}{12}\cos 3\theta - \frac{1}{30}\cos 4\theta.$$

Since $x_2'(0) = 0$, $B_2 = 0$ and

(4.17) $\quad A_2 = A\cos\theta + \frac{1}{12}\cos 3\theta - \frac{1}{30}\cos 4\theta.$

Finally, taking the coefficient of ϵ^3 in (4.13), substituting from (4.14) - (4.17), and collecting terms, we have

$$x_3'' + x_3 = u_3\cos\theta + (2A_2 + \frac{1}{12})\sin\theta + (\frac{2}{15} - A_2)\sin 3\theta$$
$$- \frac{1}{5}\sin 4\theta + \frac{1}{12}\sin 5\theta .$$

Since x_3 must be periodic, we set

(4.18) $\quad A_2 = -\frac{1}{12}$ and $u_3 = 0$.

We could obviously continue the process to obtain any desired degree of approximation. If we stop at this point, we have an approximate solution which is accurate to the order of ϵ^3. Substituting (4.12) - (4.18) into (4.11) and (4.12), we have

$$x = 2\cos\theta + \epsilon(-\frac{1}{3}\sin\theta + \frac{1}{12}\sin 3\theta)$$
$$+ \epsilon^2(-\frac{1}{16}\cos\theta + \frac{1}{12}\cos 3\theta - \frac{1}{30}\cos 4\theta) + O(\epsilon^3) ,$$

$$u = 1 - \frac{1}{16}\epsilon^2 + O(\epsilon^4),$$

and since $\theta = \omega t$, our final result is

$$x = 2\cos t\left(1 - \frac{\epsilon^2}{16}\right) + \epsilon\left[-\frac{1}{4}\sin t\left(1 - \frac{\epsilon^2}{16}\right)\right.$$

$$\left. + \frac{1}{12}\sin 3t\left(1 - \frac{\epsilon^2}{16}\right)\right] + \epsilon^2\left[-\frac{1}{16}\cos t\left(1 - \frac{\epsilon^2}{16}\right)\right.$$

$$\left. + \frac{1}{16}\cos 3t\left(1 - \frac{\epsilon^2}{16}\right) - \frac{1}{96}\cos 5t\left(1 - \frac{\epsilon^2}{16}\right)\right] + O(\epsilon^3)$$

$$= 2\cos t + \epsilon\left[-\frac{1}{4}\sin t + \frac{1}{12}\sin 3t\right]$$

$$+ \epsilon^2\left[\frac{1}{8}\sin t - \frac{1}{16}\cos t + \frac{1}{16}\cos 3t - \frac{1}{96}\cos 5t\right] + O(\epsilon^3).$$

V. Liénard's Graphical Method

In the theoretical development of the preceding two methods, we found that they are applicable to (1.1) only if the nonlinear term is small. If the nonlinear term is large, Liénard's method, [5], furnishes a practical method of obtaining an approximate solution. This is a graphical construction for the field directions of the solution curves in the x,v plane, where we recall $v = \dot{x}$. The solution curves themselves are approximated by the construction of tangent elements from point to point. Theoretically, any desired degree of accuracy can be obtained by utilizing short enough line segments.

We will consider the equation to be in the form

$$\ddot{x} + \epsilon f(x,\dot{x}) + x = 0,$$

and we will assume that $\dot{x} = v$ is a separate variable, in order to obtain

(6.1) $$\frac{dv}{dx} = -\frac{\epsilon f(x,v) + x}{v}.$$

In preparation, we plot the curve C, whose equation is $x = -\epsilon f(x,v)$, Figure 3. We then obtain the field direction vector s at any point P in the following manner:

1) Mark the point Q on C which has the same ordinate as P.
2) Mark the point R on the x axis which has the same abscissa as Q.
3) s is perpendicular to RP and has clockwise sense.

To justify this construction, we see that for $P(x_1,v_1)$, QP has length $x_1 - [-\epsilon f(x_0 v_1)] = x_1 + f(x_1,v_1)$ and RQ has length v_1, so that RP has slope $v_1 / [x_1 + \epsilon f(x_1 v_1)]$. The perpendicular to RP will then have slope $-[x_1 + \epsilon f(x_1,v_1)]/v_1$, which is exactly $\frac{dv}{dx}$ at the point x_1,v_1.

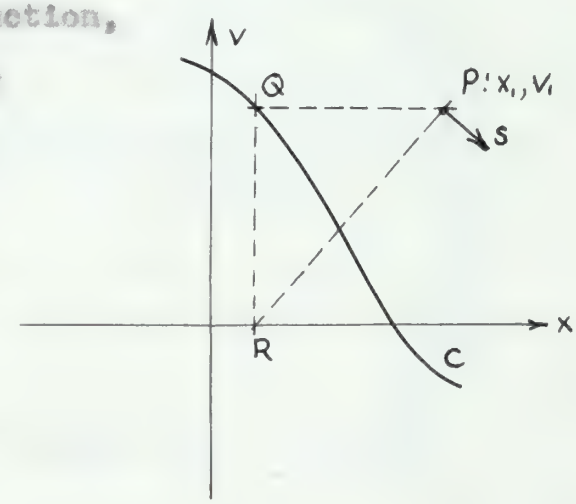

Fig. 2 Liénard's Construction.

In Figure 4, Liénard's construction has been applied to the van der Pol equation for the case $\epsilon = 2$. Two approximate solution curves and the curve C, $x = 2(v - \frac{1}{3}v^3)$, are shown. We see that solution 1, which begins near the

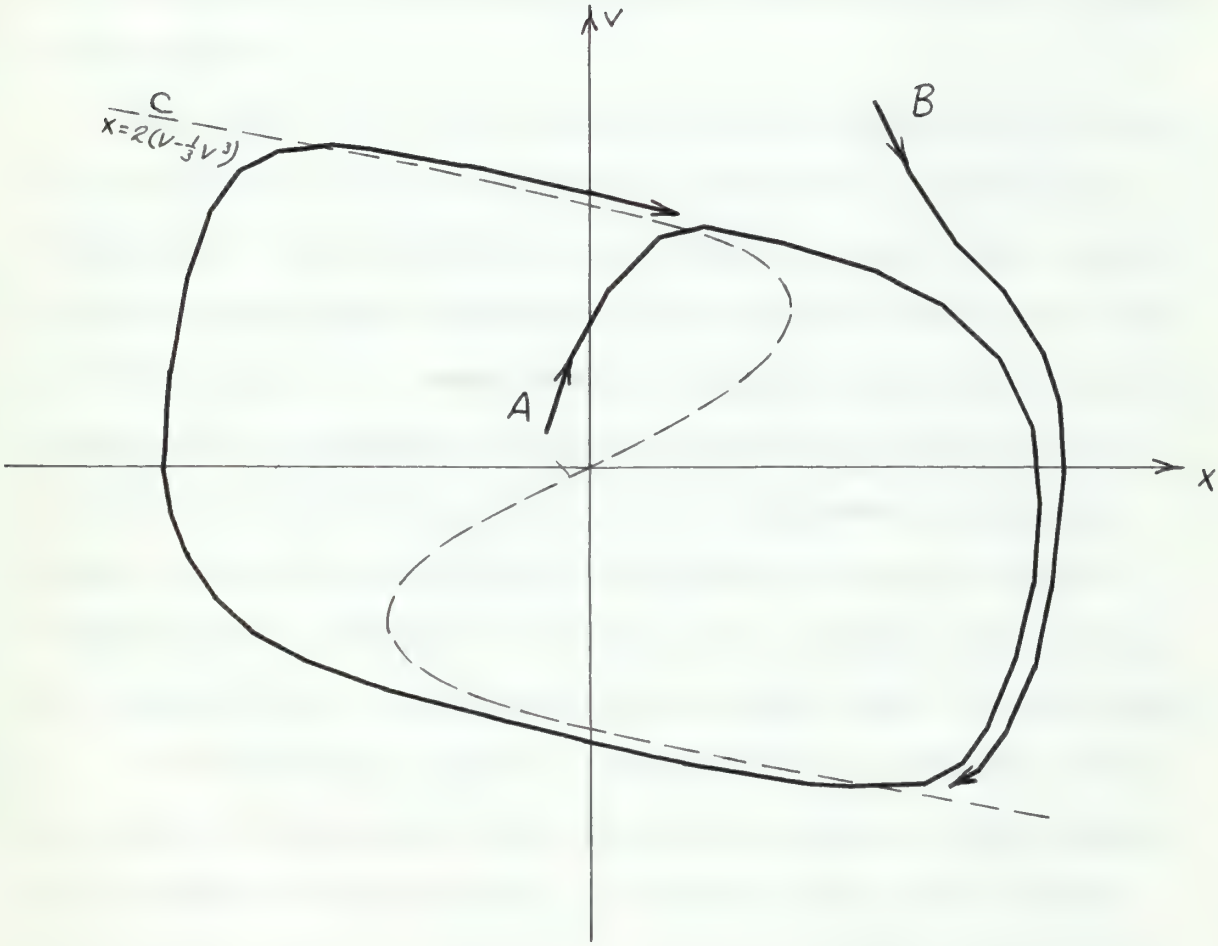

Fig. 4 Solution curves for van der Pol's equation for $\epsilon = 3$.

origin, spirals outward to approach a closed solution, and that solution S, which begins outside this closed solution, spirals inward to approach it.

Once we have obtained the solution curve, we can construct an approximation of $x = x(t)$ by using $v(x) = \dot{x}(x)$ as the slope at each displacement x and constructing tangent line segments.

We shall give the description and a simplified diagram of a machine designed to perform Liénard's construction. Because of its simplicity, it is probable that this machine has been previously conceived and possibly constructed; but the wide variety of publications in which analogue computing machinery is discussed, make determination if its originality very difficult. Hartree [6] discusses a general class of machines which use the slope of the solution to determine the direction of travel of a tracing wheel, but he does not give a method of setting the wheel to the slope.

This machine, Figure 6, consists of a plotting table, an dealer sharp-edged wheel driven by either a small motor or a spring and escapement, a pivoted guide which establishes the slope of the wheel, and two plastic blades with scribed reference lines which position the guide. The plastic blades are positioned manually by two operators or by electronic curve followers. The sharp-edged wheel is mounted with its axis passing through the pivoted guide and is taked by a wick so that it traces the solution curves as it moves.

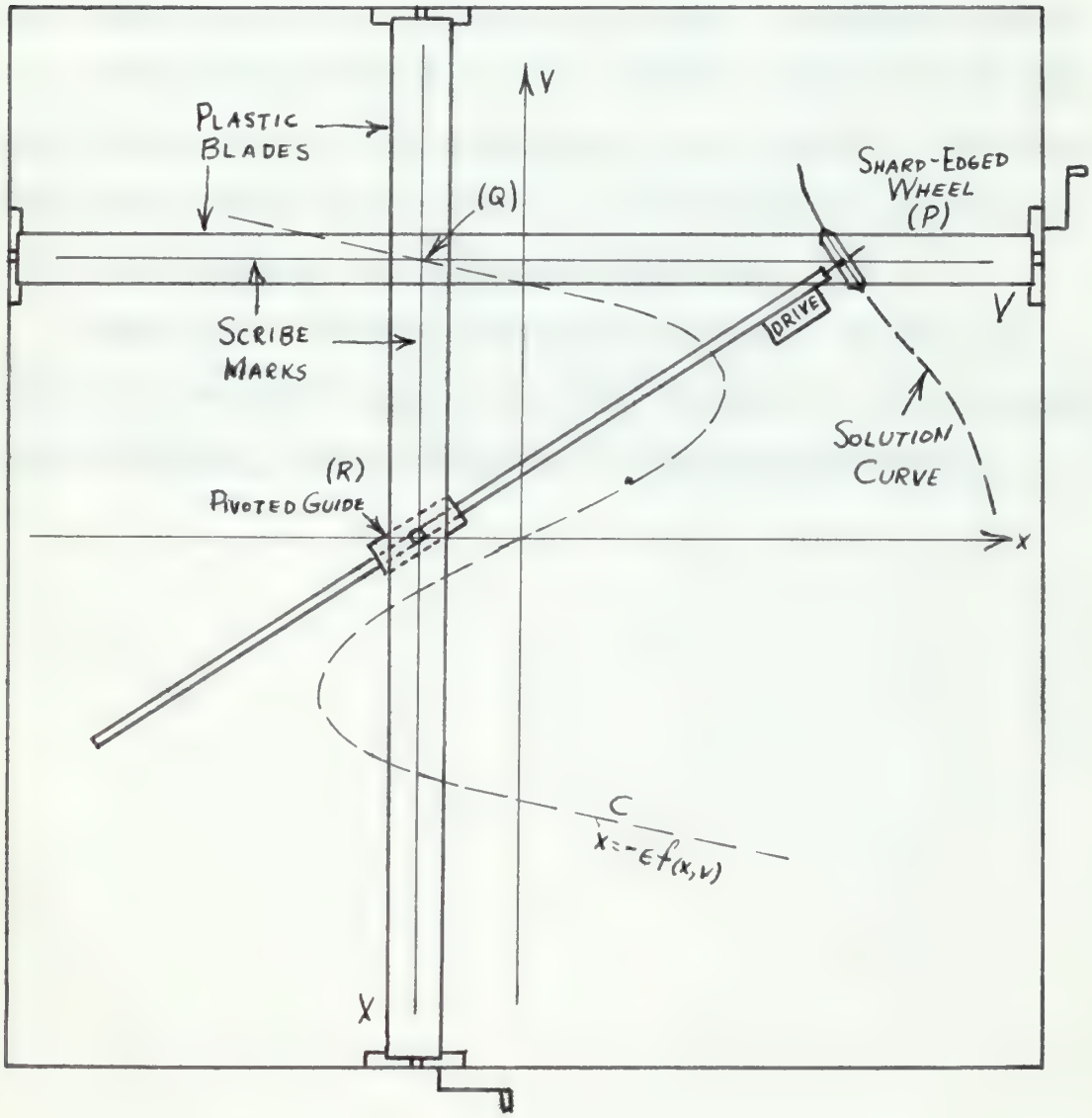

Fig. 3 Machine to perform Liénard's construction.

The curve G, $s = -\epsilon[f(s,v)]$, is plotted on a paper which is mounted on the table. If we consider Figure 7, the wheel corresponds to point P; the y blade is kept coincident with a reference point mounted above the wheel and determines point C on curve G. The Y blade is kept coincident with the intersection of the Y blade and curve G and determines point E on the x axis. The pivot of the guide is mounted on the X blade at the x axis. Since the wheel moves about the pivot V, it will move perpendicular to DF and generate the direction vector z.

Since this machine determines the slope of the solution curve continuously, it will generate a theoretically exact solution, rather than the polygon approximation.

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